EXERCISES [MAI 5.16]

AREAS AND VOLUMES

SOLUTIONS

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A. Paper 1 questions (SHORT)

AREAS

1. (a) (i) $\int_0^{\frac{\pi}{2}} \cos x dx = 1$, `(ii) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx = -2$ `(iii) $\int_0^{\frac{3\pi}{2}} \cos x dx = -1$

(b) (i) 1 (ii) 2 (iii) 3

2. Area = $\int_0^{\frac{3\pi}{4}} \sin x\, dx = [-\cos x]_0^{\frac{3\pi}{4}} = \left(-\cos\frac{3\pi}{4}\right) - (-\cos 0) = \frac{\sqrt{2}}{2} + 1$

3. (i) $2x + \ln(x-1) + c$

(ii) $A = \int_2^4 f(x)dx = \int_2^4 \left(2 + \frac{1}{x-1}\right)dx = \left[2x + \ln(x-1)\right]_2^4 = (8 + \ln 3) - (4 + \ln 1) = 4 + \ln 3$

4. $p = 3$

Area = $\int_0^{\frac{\pi}{2}} 3\cos x dx = [3\sin x]_0^{\frac{\pi}{2}} = 3$ square units

5. (a) Area of A = $\int_{\frac{3\pi}{2}}^{2\pi} \cos x dx = 1$

(b) $\int_{\frac{4\pi}{3}}^{\frac{3\pi}{2}} y\, dx = -\frac{2-\sqrt{3}}{2}$ (– 0.134)

Area of B = $\frac{2-\sqrt{3}}{2}$ (0.134)

Total Area = $\frac{4-\sqrt{3}}{2}$ (i.e. 1 + 0.134 = 1.13

6. (a) $\int (1 + 3\sin(x+2))dx = x - 3\cos(x+2) + c$

(b) (i) $1 + 3\sin(x + 2) = 0 \Rightarrow x = 1.48$

$a = 1.48$

(ii) 6.26

7. Area = $\int_0^{k} \sin 2x dx = 0.85$

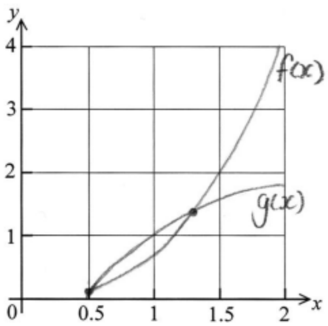
$\int_0^{k} \sin 2x dx = \left[\frac{-1}{2}\cos 2x\right]_0^k = \frac{-1}{2}\cos 2k + 0.5$

$\frac{-1}{2}\cos 2k + 0.5 = 0.85$ ($\Leftrightarrow \cos 2k = -0.7$)

THEN $k = 1.17$

8. $\int_1^a \frac{1}{x}\,dx = 2 \Rightarrow [\ln x]_1^a = 2 \Rightarrow \ln a = 2 \Rightarrow a = e^2$

9. (a)



(b) $A = \int_a^b g(x) - f(x)\,dx$

$a = 0.50546\ldots,\ b = 1.227\ldots$

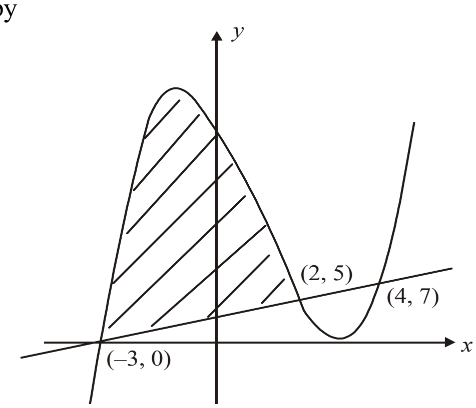
$A = 0.201$

10. x-intercepts are = π, 2π, 3π.

Area = $\left|\int_{\pi}^{2\pi} \frac{\sin x}{x}\,dx\right| + \int_{2\pi}^{3\pi} \frac{\sin x}{x}\,dx = 0.4338 + 0.2566 = 0.690$ units^2

OR directly by GDC Area = 0.690 units^2

11. Region required is given by



METHOD 1

outer intersections are at $x = -3$ and $x = 4$

$\Rightarrow$Area = $\int_{-3}^{4} |y_1 - y_2|\,dx = 101.75$

METHOD 2

intersections are at $x = -3$, $x = 2$, $x = 4$

Area = $\int_{3}^{2} \left(x^3 - 3x^2 - 9x + 27 - (x+3)\right)dx + \int_2^4 \left(x + 3 - (x^3 - 3x^2 - 9x + 27)\right)dx = 101.75$

12.

Region enclosed by	Expression for the area	Area
$f(x) = \cos(x^2)$ and $g(x) = e^x$, for $-1.5 \le x \le 0.5$.	$\int_{-1.11}^{0} \cos x^2 - e^x \, dx$	0.282
$y = \sin x$ and $y = x^2 - 2x + 1.5$, for $0 \le x \le \pi$.	$\int_{0.6617}^{1.7010} (\sin x - (x^2 - 2x + 1.5))\, dx$	0.271
$y = \ln x$ and $y = e^x - e$, for $x > 0$.	$\int_{0.233}^{1} (\ln x - e^x + e) dx$	0.201
$y = \dfrac{2}{1+x^2}$ and $y = e^{x/3}$, for $-3 \le x \le 3$.	$\int_{-1.5247}^{0.74757} \left(\dfrac{2}{1+x^2} - e^{\frac{x}{3}} \right) dx =$	1.22
$f(x) = 4 - x^2$ and $g(x) = (x+1)\cos x$	$\int_{-1.92}^{2.72} (4 - x^2 - (x+1)\cos x) dx$	9.41
$y = e^{-x} - x + 1$ and the coordinate axes	$\int_{0}^{1.278} (e^{-x} - x + 1) dx$	1.18

13. (a) 2.31

(b) (i) 1.02 (ii) 2.59

(c) $\int_p^q f(x) dx = 9.96$

split into two regions, make the area below the x-axis positive

14. (a) π (3.14)

(b) (i) $\int_0^{\pi} e^x \sin x \, dx$

(ii) Area = 12.1

15. (a) (i) $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

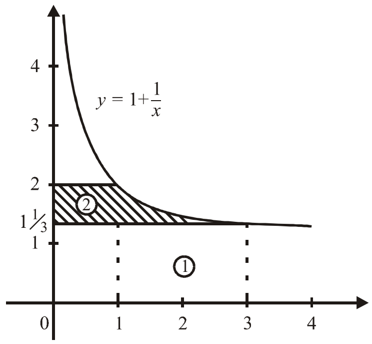
therefore $\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) = 0$

(ii) $\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1$

$x = \frac{3\pi}{4}$

(b) Required area = $\int_0^{\frac{3\pi}{4}} e^x (\cos x + \sin x)\, dx = 7.46$ sq units

16.



METHOD A (quickest!) **(in terms of y)**

Area = $\int_{1\frac{1}{3}}^{2} x\,dy = \int_{1\frac{1}{3}}^{2} \frac{1}{(y-1)}\,dy = \left[\ln(y-1)\right]_{1\frac{1}{3}}^{2} = \ln 1 - \ln\frac{1}{3} = \ln 3$

METHOD B

Area from $x = 1$ to $x = 3$, $A = \int_1^3 \left(1+\frac{1}{x}\right)dx = [x + \ln x]_1^3$

$= (3 + \ln 3) - (1 + \ln 1) = 2 + \ln 3$

Area rectangle ① $= 2 \times 1\frac{1}{3} = 2\frac{2}{3}$, area rectangle ② $= 1 \times \frac{2}{3} = \frac{2}{3}$

Shaded area $= 2 + \ln 3 - 2\frac{2}{3} + \frac{2}{3} = \ln 3$

17. (a) (i) $f'(x) = \frac{-3(x^2-1)-(-3x)(2x)}{(x^2-1)^2} = \frac{-3x^2+3+6x^2}{(x^2-1)^2} = \frac{3x^2+3}{(x^2-1)^2} = \frac{3(x^2+1)}{(x^2-1)^2}$

(b) Area = $\int_0^a g(x)\,dx = \int_0^a \frac{3x^2+3}{(x^2-1)^2}\,dx = \left[2 - \frac{3x}{x^2-1}\right]_0^a = \left[2 - \frac{3a}{a^2-1}\right] - [2-0] = -\frac{3a}{a^2-1}$

Area = 2 $\Leftrightarrow -\frac{3a}{a^2-1} = 2 \Leftrightarrow 2a^2 + 3a - 2 = 0 \Leftrightarrow a = \frac{1}{2}$ $\quad a = -2$ (rejected)

18. (a) $\frac{32}{3}(\sqrt{2}-1) \cong 4.42$

(b) 10.7

VOLUMES

19. (a) $V = \int_a^b \pi y^2 dx \; \pi\int_0^2 \left(2x - x^2\right)^2 dx$

(b) Volume = $\pi\int_0^2 \left(4x^2 - 4x^3 + x^4\right)dx = \pi\left[4\frac{x^3}{3} - 4\frac{x^4}{4} + \frac{x^5}{5}\right]_0^2 = \frac{16\pi}{15}$

OR directly by GDC: Volume = 3.35 $\left(= \frac{16\pi}{15} = 1.07\pi\right)$

20. $\int \pi y^2\,dx = \pi\int\left(x^{\frac{1}{2}}\right)^2 dx = \frac{x^2}{2}$ $\qquad V = \pi\left[\frac{x^2}{2}\right]_0^a = \frac{\pi a^2}{2}$

$\frac{1}{2}\pi a^2 = 0.845\pi \Leftrightarrow a^2 = 1.69 \Leftrightarrow a = 1.3$

21. (a) At $x = 0$ $y = e^0 = 1$ $P(0, 1)$

(b) $V = \pi \int_0^{\ln 2} (e^{x/2})^2 \,dx$

(c) $V = \int_0^{\ln 2} e^x dx = \pi[e^x]_0^{\ln 2} = \pi[e^{\ln 2} - e^0] = \pi[2 - 1] = \pi$

22. (a) $\int 3\sin^2 x \cos x dx = \sin^3 x + c$

(b) $V = \int_a^b \pi y^2 dx = \int_0^{\frac{\pi}{2}} \pi \left(\sqrt{3} \sin x \cos^{\frac{1}{2}} x \right)^2 dx = \pi \int_0^{\frac{\pi}{2}} 3 \sin^2 x \cos x \,dx$

$= \pi \left[\sin^3 x \right]_0^{\frac{\pi}{2}} = \pi(1 - 0) = \pi$

23. (i) $V = \pi \int_1^{1.5} f(x)^2 \,dx$

(ii) $V = 105$ (or 33.3π)

24. $V = \pi \int_0^{\frac{\pi}{4}} (\sin 3x)^2 \,dx = 0.476\pi = 1.4954... \cong 1.50$

25. (a) $A = \int_1^3 \frac{\ln x}{x^3} dx = 0.161$

(b) $V = \pi \int_1^3 \left(\frac{\ln x}{x^3} \right)^2 dx = 0.0458$

26. $V = \pi \int_0^k e^{2x} dx = \frac{\pi}{2}[e^{2x}]_0^k = \frac{\pi}{2}(e^{2k} - 1)$

27. $V = \pi \int_0^{2a} y^2 dx = \pi \int_0^{2a} 8a(2a - x)dx = 8\pi a \int_0^{2a} (2a - x)dx$

$= 8\pi a \left[2ax - \frac{x^2}{2} \right]_0^{2a}$

$= 8\pi a (4a^2 - 2a^2)$

$= 16\pi a^3$

28. (a) $A = \int_0^a (ax + 2) - (x^2 + 2)dx = \int_0^a (ax - x^2)dx = \left[a\frac{x^2}{2} - \frac{x^3}{3} \right]_0^a = \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}$ units^2

(b) $V_x = \pi \int_0^a \left((ax + 2)^2 - (x^2 + 2)^2 \right) dx = \pi \int_0^a (a^2x^2 + 4ax + 4 - x^4 - 4x^2 - 4)dx$

$= \pi \left[\frac{1}{3} a^2 x^3 + 2ax^2 - \frac{1}{5} x^5 - \frac{4}{3} x^3 \right]_0^a = \pi \left(\frac{2}{15} a^5 + \frac{2}{3} a^3 \right)$ units^3

(c) $V_y = \pi \int_2^{a^2+2} [(y - 2) - (\frac{y-2}{a})^2] dy$

B. Paper 2 questions (LONG)

29. (a) $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

when $x = e$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

tangent line: $y - 1 = \left(\dfrac{1}{e}\right)(x - e) \Rightarrow y = \dfrac{x}{e}$

$x = 0 \Rightarrow y = \dfrac{0}{e} = 0$ Thud (0, 0) is on line

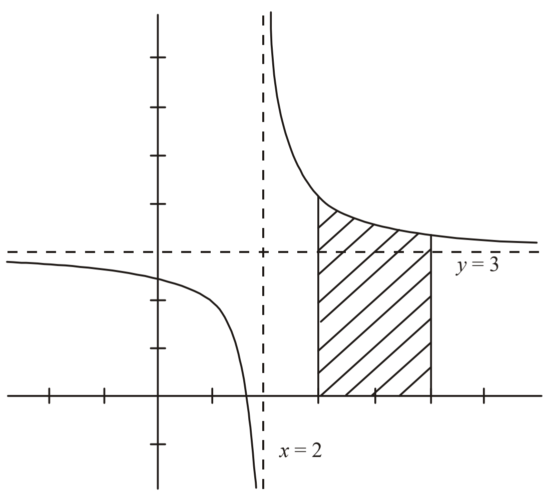
(b) $\dfrac{d}{dx}(x \ln x - x) = (1) \times \ln x + x \times \left(\dfrac{1}{x}\right) - 1 = \ln x$

(c) Area = area of triangle – area under curve

$= \left(\dfrac{1}{2} \times e \times 1\right) - \int_1^e \ln x dx = \dfrac{e}{2} - [x \ln x - x]_1^e = \dfrac{e}{2} - \{(e\ln e - 1 \ln 1) - (e - 1)\}$

$= \dfrac{e}{2} - \{e - 0 - e + 1\} = \dfrac{1}{2}e - 1.$

30. (a) (i)



(ii) (Vertical asymptote) $x = 2$, (Horizontal asymptote) $y = 3$

(b) (i) $3x + \ln(x - 2) + C$

(ii) $[3x + \ln(x-2)]_3^5 = (15 + \ln 3) - (9 + \ln 1) = 6 + \ln 3$

(c) See graph

31. (a) (i) intersection points $x = 3.77$, $x = 8.30$

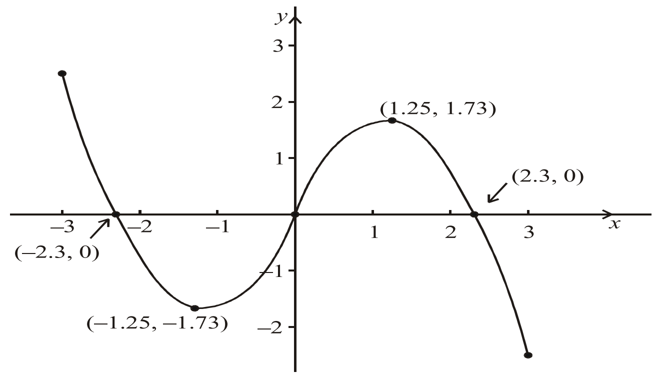
$$\int_{3.77}^{8.30} g(x)dx - \int_{3.77}^{8.30} f(x)dx = \int_{3.77}^{8.30} \left((-4\cos(0.5x) + 2) - (\ln(3x-2) + 1)\right)dx$$

(ii) $A = 6.46$

(b) (i) $f'(x) = \dfrac{3}{3x-2}$ (ii) $g'(x) = 2\sin(0.5x)$

(c) $f'(x) = g'(x) \Leftrightarrow x = 1.43, x = 6.10$

32. (a)

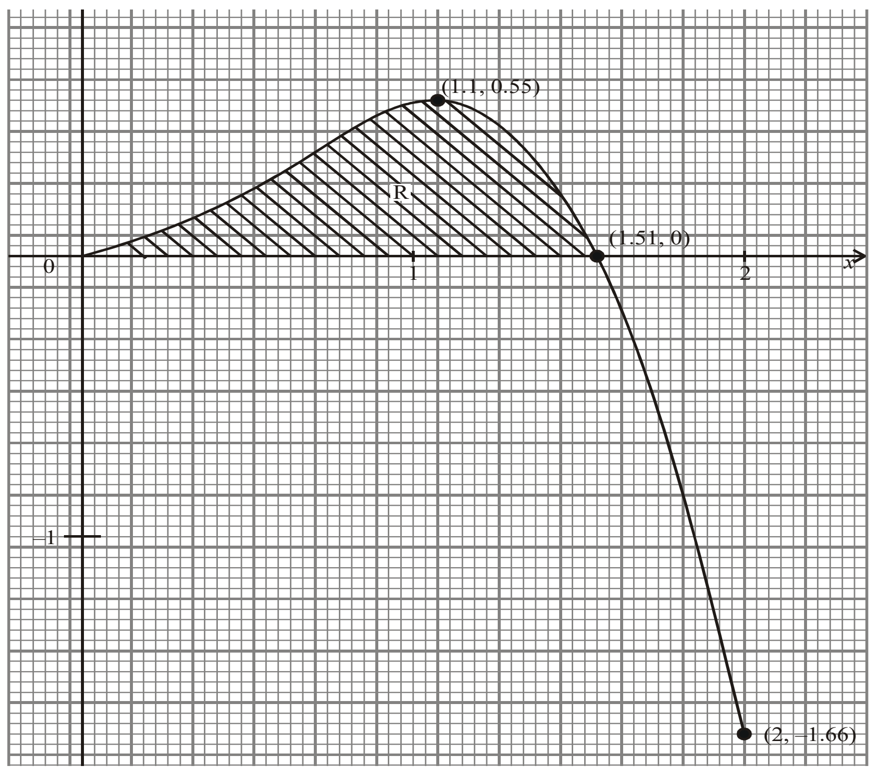


(b) $x = 2.31$

(c) $\int (\pi \sin x - x)\mathrm{d}x = -\pi \cos x - \frac{x^2}{2} + C$

Required area = $\int_0^1 (\pi \sin x - x)\mathrm{d}x = 0.944$

33. (a)(i)&(c)(i)



(ii) Approximate positions are
positive x-intercept (1.57,0), max point (1.1,0.55), end points (0, 0) and (2,–1.66)

(b) $x^2 \cos x = 0 \qquad x \neq 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

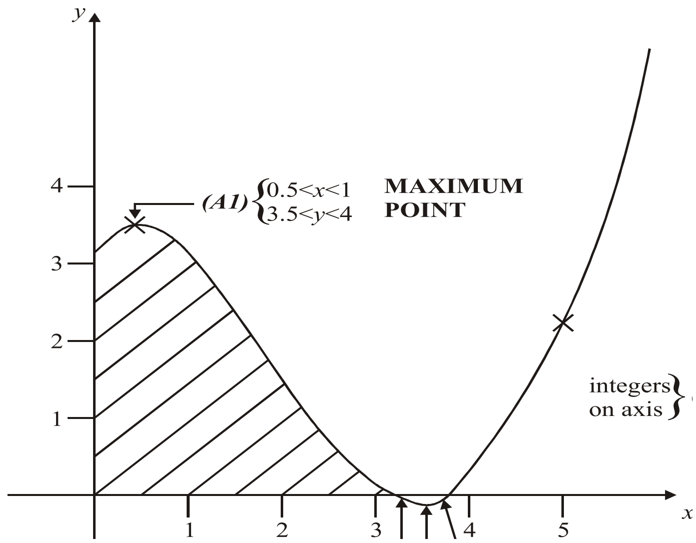
(c) (i) see graph (ii) $\int_0^{\pi/2} x^2 \cos x \, \mathrm{d}x$

(d) Integral = 0.467 by GDC **OR**

Integral = $\left[x^2 \sin x + 2x \cos x - 2 \sin x\right]_0^{\pi/2} = \left[\frac{\pi^2}{4}(1) + 2\left(\frac{\pi}{2}\right)(0) - 2(1)\right] - [0 + 0 - 0]$

$= \frac{\pi}{2} - 2$ (exact) **or** 0.467 (3 s.f.)

34. (a)



(b) π is a solution if and only if $\pi + \pi\cos\pi = 0$.
Now $\pi + \pi\cos\pi = \pi + \pi(-1) = 0$

(c) $x = 3.696\ 722\ 9... = 3.69672$ (6s.f.)

(d) See graph: $\int_0^{\pi}(\pi + x\cos x)\mathrm{d}x$

(e) **EITHER** $\int_0^{\pi}(\pi + x\cos x)\mathrm{d}x = 7.86960$ (6 s.f.)

35. (a) From graph, period = 2π

(b) Range : $-0.4 < y < 0.4$

(c) (i) $f'(x) = \cos x\,(2\sin x\cos x) - \sin x\,(\sin x)^2 = \sin x\{2\cos^2 x - \sin^2 x\}$

(ii) $f'(x) = 0 \Rightarrow \sin x\{2\cos^2 x - \sin^2 x\} = 0 \Rightarrow \sin x\{3\cos^2 x - 1\} = 0$

$\Rightarrow 3\cos^2 x - 1 = 0 \Rightarrow \cos x = \pm\sqrt{\left(\frac{1}{3}\right)}$

At A, $f(x) > 0$, hence $\cos x = \sqrt{\left(\frac{1}{3}\right)}$

(iii) $f(x) = \sqrt{\left(\frac{1}{3}\right)\left(1 - \left(\sqrt{\left(\frac{1}{3}\right)}\right)^2\right)} = \frac{2}{3} \times \frac{1}{\sqrt{3}} = \frac{2}{9}\sqrt{3}$

(d) $x = \frac{\pi}{2}$

(e) (i) $\int (\cos x)(\sin x)^2\,\mathrm{d}x = \frac{1}{3}\sin^3 x + c$

(ii) Area = $\int_0^{\pi/2}(\cos x)(\sin x)^2\,\mathrm{d}x = \frac{1}{3}\left\{\left(\sin\frac{\pi}{2}\right)^3 - (\sin 0)^3\right\} = \frac{1}{3}$

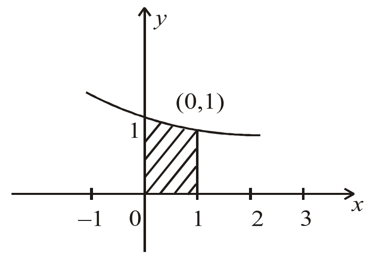
(f) At C $f''(x) = 0 \Leftrightarrow 9\cos^3 x - 7\cos x = 0 \Leftrightarrow \cos x(9\cos^2 x - 7) = 0$

$\Rightarrow x = \frac{\pi}{2}$ (reject) **or** $x = \cos^{-1}\frac{\sqrt{7}}{3} = 0.491$ (3 s.f.)

36. (a) $\int_0^1 e^{-kx}dx = \left[-\frac{1}{k}e^{-kx}\right]_0^1 = -\frac{1}{k}(e^{-k} - e^0) = -\frac{1}{k}(e^{-k} - 1) = -\frac{1}{k}(1 - e^{-k})$

(b) $k = 0.5$

(i)



(ii) Shading (see graph)

(iii) Area = $\int_0^1 e^{-kx}dx = \frac{1}{0.5}(1 - e^{05})$ **OR** Area = 0.787 (3 s.f.)

(c) Volume = $\pi\int_0^1 e^{-2kx}]dx = \pi\left[\frac{e^{-2kx}}{-2k}\right]_0^1 = \pi\left[\frac{e^{-2k}}{-2k} - \frac{1}{-2k}\right] = \frac{\pi}{2k}(1 - e^{-2k})$

37. (a) At A, $x = 0 \Rightarrow y = \sin(e^0) = \sin(1) \Rightarrow$ coordinates of A = (0,0.841)

(b) $\sin(e^x) = 0 \Rightarrow e^x = \pi \Rightarrow x = \ln\pi$ (or $k = \pi$)

(c) (i) Maximum value of sin function = 1

(ii) $\frac{dy}{dx} = e^x\cos(e^x)$

$\frac{dy}{dx} = 0 \Rightarrow e^x\cos(e^x) = 0 \Rightarrow e^x = 0$ (impossible) or $\cos(e^x) = 0$

$\Rightarrow e^x = \frac{\pi}{2} \Rightarrow x = \ln\frac{\pi}{2}$

(d) (Area = $\int_0^{\ln\pi}\sin(e^x)dx$ = 0.90585 = 0.906 (3 s.f.)

38. (a) $a = 1 - \pi \quad b = 1 + \pi$

(b) (i) $\int_{-2.14}^{1}h(x)dx - \int_1^2 h(x)dx$ **OR** $\int_{-2.14}^{1}h(x)dx + \left|\int_1^2 h(x)dx\right|$

(ii) $5.141... - (-0.1585...) = 5.30$

(c) (i) $y = 0.973$ (ii) $-0.240 < k < 0.973$

39. (a) intersection points at $x = -1$ and $x = 1$, Area = $\int_{-1}^{1}e^x(1 - x^2)dx$

(b) $f(0) = 1$. Thus P(0,1)

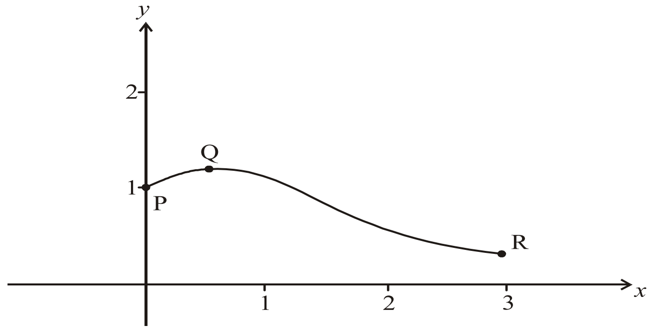
(c) $f'(0) = 1$, gradient of the normal = -1

$y - 1 = -1(x - 0) \Leftrightarrow y - 1 = -x \Leftrightarrow y = -x + 1 \Leftrightarrow x + y = 1$

(d) (i) intersection points at $x = 0$ and $x = 1$, Area = $\int_0^1\left(e^x(1 - x^2) - (1 - x)\right)dx$

(ii) area $R = 0.5$

40. (a)



(b) (i) $f'(x) = 2e^{-x} + (2x + 1)(-e^{-x}) = (1 - 2x)e^{-x}$

(ii) At **Q**, $f'(x) = 0$

$x = 0.5, y = 2e^{-0.5}$ **Q** is $(0.5, 2e^{-0.5})$

(c) $1 \le k < 2e^{-0.5}$

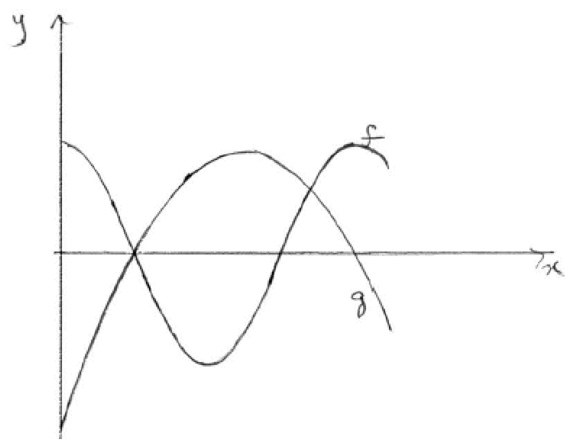
(d) At R, $y = 7e^{-3}$ (= 0.34850 ...)

Gradient of (PR) is $\frac{7e^{-3} - 1}{3}$ $(= -0.2172)$

Equation of (PR) is $y = \left(\frac{7e^{-3} - 1}{3}\right)x + 1$ OR $y = -0.2172x + 1$

Area is $\int_0^3 \left((2x+1)e^{-x} - \left(\frac{7e^{-3} - 1}{3}x + 1\right)\right)dx = 0.529$

41. (a)



(b) (i) (2,0) (accept $x = 2$)

(ii) period = 8

(iii) amplitude = 5

(c) (i) (2, 0), (8, 0) (accept $x = 2, x = 8$)

(ii) $x = 5$ (must be an equation)

(d) intersect when $x = 2$ and $x = 6.79$

area = $\int_2^{6.79} \left((-0.5x^2 + 5x - 8 - \left(5\cos\frac{\pi}{4}x\right)\right)$ = 27.6

42. (a) $f'(x)=\dfrac{1}{2\sqrt{x}}$ $f'(4)=\dfrac{1}{4}$

gradient of normal = – 4

normal; $y-2=-4(x-4) \Rightarrow y=-4x+18$

(b) $-4x+18=0 \Rightarrow x=\dfrac{18}{4}\left(=\dfrac{9}{2}\right)$

(c) area of $R=\int_0^4 \sqrt{x}\,\mathrm{d}x+\int_4^{4.5}(-4x+18)\,\mathrm{d}x$, **OR** $\int_0^4 \sqrt{x}\,\mathrm{d}x+\dfrac{1}{2}\times 0.5\times 2$ (triangle)

(d) $R=\left[\dfrac{2}{3}x^{\frac{3}{2}}\right]_0^4+0.5=\dfrac{2}{3}4^{\frac{3}{2}}+\dfrac{1}{2}=\dfrac{16}{3}+\dfrac{1}{2}=\dfrac{35}{6}$

43. (a) $f''(x)=0 \Leftrightarrow 2ax(x^2-3)=0 \Leftrightarrow x=0$ or $x^2=3$

$(0, 0), \left(\sqrt{3},\dfrac{a\sqrt{3}}{4}\right),\left(-\sqrt{3},-\dfrac{a\sqrt{3}}{4}\right)$

(b) (i) area $=\left[\dfrac{a}{2}\ln(x^2+1)\right]_3^7=\dfrac{a}{2}(\ln 50-\ln 10)=\dfrac{a}{2}\ln 5$

(ii) **METHOD 1**

the shift does not change the area: $\int_4^8 f(x-1)dx=\int_3^7 f(x)\mathrm{d}x, \dfrac{a}{2}\ln 5$

the factor of 2 doubles the area: $\int_4^8 2f(x-1)\mathrm{d}x=2\int_4^8 f(x-1)\mathrm{d}x$ $\left(=2\int_3^7 f(x)\mathrm{d}x\right)$

$\int_4^8 2f(x-1)\mathrm{d}x=a\ln 5$

METHOD 2

changing variable: let $w=x-1$, so $\dfrac{\mathrm{d}w}{\mathrm{d}x}=1$

Integral = $2\int f(w)\mathrm{d}w=\dfrac{2a}{2}\ln(w^2+1)+c$

Limits: when $x=4 \Rightarrow w=3$, when $x=8 \Rightarrow w=7$,
$\int_4^8 2f(x-1)\mathrm{d}x=\left[a\ln(w^2+1)\right]_3^7=a\ln 50-a\ln 10=a\ln 5$

44. (a) $f(x)=0 \Leftrightarrow a=-1.73, b=1.73$ $(a=-\sqrt{3}, b=\sqrt{3})$

(b) **EITHER** setting $f'(x)=0$, **OR** directly by GDC $c=1.15$

(c) finding 2 areas

$-\int_{-1.73...}^{0} f(x)\mathrm{d}x+\int_0^{1.149...} f(x)\mathrm{d}x$

area = 2.07 (accept 2.06)

45. (a) (i) $\sin x = 0 \Leftrightarrow x = 0, x = \pi$

(ii) $\sin x = -1 \Leftrightarrow x = \dfrac{3\pi}{2}$

(b) $\dfrac{3\pi}{2}$

(c) $k = \int_0^{\frac{3\pi}{2}} (6 + 6\sin x)\mathrm{d}x = [\ 6x - 6\cos x\ ]_0^{3\pi/2}$

$= 6\left(\dfrac{3\pi}{2}\right) - 6\cos\left(\dfrac{3\pi}{2}\right) - (-6\cos 0)$

$k = 9\pi + 6$

(d) translation of $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$

(e) the area under g is the same as the shaded region in f

$p = \dfrac{\pi}{2}, p = 0$

46. (a) $13 = A\mathrm{e}^0 + 3 \Leftrightarrow 13 = A + 3 \Leftrightarrow A = 10$

(b) $f(15) = 3.49 \Leftrightarrow 3.49 = 10\mathrm{e}^{15k} + 3 \Leftrightarrow k = -0.201$

(c) (i) $f(x) = 10\mathrm{e}^{-0.201x} + 3$

$f'(x) = 10\mathrm{e}^{-0.201x} \times -0.201 = -2.01\mathrm{e}^{-0.201x}$

(ii) $f'(x) < 0$, derivative always negative

(iii) $y = 3$

(d) finding limits 3.8953…, 8.6940…

Area = $\int_{3.90}^{8.69} g(x) - f(x)\mathrm{d}x, \int_{3.90}^{8.69} [(-x^2 + 12x - 24) - (10\mathrm{e}^{-0.201x} + 3)]\mathrm{d}x = 19.5$

47. (a) (i) $a = 1.3302... \cong 1.33$

(ii) $p = (10x + 2) - (1 + \mathrm{e}^{2x}) = 10x - \mathrm{e}^{2x} + 1$

(iii) $\dfrac{\mathrm{d}p}{\mathrm{d}x} = 10 - 2\mathrm{e}^{2x}$

$\dfrac{\mathrm{d}p}{\mathrm{d}x} = 0 \Leftrightarrow 10 - 2\mathrm{e}^{2x} = 0 \Leftrightarrow \mathrm{e}^{2x} = 5 \Leftrightarrow x = \dfrac{\ln 5}{2}$ (= 0.805)

(iv) $p_{\max} = 5\ln 5 - 5 + 1 = 5\ln 5 - 4$

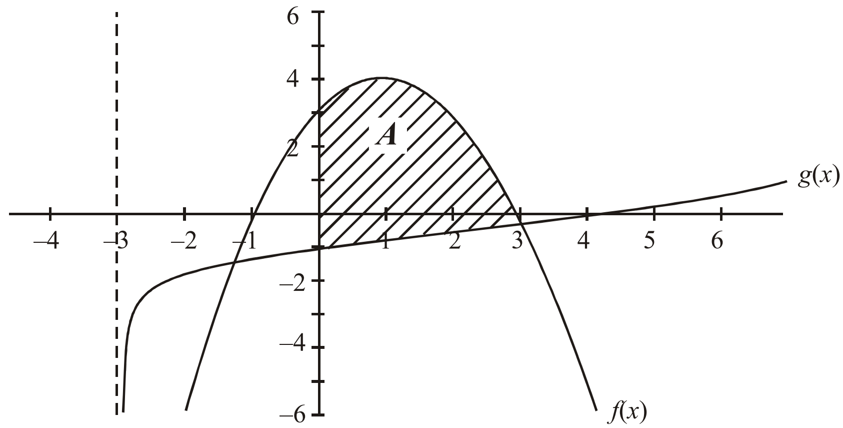
(b) $A = \int_0^{1.3302} \left[(10x + 2) - (1 + e^{2x})\right] dx = \int_0^{1.3302} (10x - e^{2x} + 1) dx$

$A \cong 3.53$

(c) $V = \pi \int_0^{1.3302} \left[(10x + 2)^2 - (1 + e^{2x})^2\right] dx = 168.5369... \cong 169$

$V \cong 169$

48. (a)



(b) (i) $x = -3$ is the vertical asymptote.

(ii) x-intercept: $x = 4.39$ ($= e^2 - 3$), y-intercept: $y = -0.901$ ($= \ln 3 - 2$)

(c) $f(x) = g(x) \Leftrightarrow x = -1.34$ or $x = 3.05$

(d) (i) See graph

(ii) Area of $A = \int_0^{3.05} \left(4 - (1-x)^2\right) - (\ln(x+3) - 2)\, dx$

(iii) Area of $A = 10.6$

(e) $D = f(x) - g(x) = 5 + 2x - x^2 - \ln(x+3)$

$$\frac{dD}{dx} = 2 - 2x - \frac{1}{x+3}$$

Maximum occurs when $\frac{dD}{dx} = 0 \Leftrightarrow 2 - 2x = \frac{1}{x+3} \Leftrightarrow 5 - 4x - 2x^2 = 0 \Leftrightarrow x = 0.871$

Then $D = 4.63$

OR directly by GDC

Maximum of $D = f(x) - g(x)$ occurs at $x = 0.871$ and the maximum value is 4.63.

49. (a) (i) $f'(x) = \dfrac{x\frac{1}{x} - \ln x \times 1}{x^2} = \dfrac{1 - \ln x}{x^2}$

$f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4} = \dfrac{2\ln x - 3}{x^3}$

Stationary point where $f'(x) = 0 \Leftrightarrow \ln x = 1 \Leftrightarrow x = e$

$f''(e) < 0$ so maximum.

(ii) Exact coordinates $x = e$, $y = \dfrac{1}{e}$

(b) Solving $f''(x) = 0 \Leftrightarrow \ln x = \dfrac{3}{2} \Leftrightarrow x = e^{\frac{3}{2}}$ (4.48)

(c) Area $= \int_1^5 \dfrac{\ln x}{x}\, dx$ indefinite integral by substitution or inspection: $\dfrac{(\ln x)^2}{2} + c$

Area $= \left[\dfrac{(\ln x)^2}{2}\right]_1^5 = \dfrac{1}{2}\left((\ln 5)^2 - (\ln 1)^2\right) = \dfrac{1}{2}(\ln 5)^2$ (= 1.30)

(d) $V = \int_1^5 \pi\left(\dfrac{\ln x}{x}\right)^2 dx = 1.38$

50. (a) (i) gradient $= \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{f(a) - 0}{a - \frac{2}{3}} = \dfrac{a^3 - 0}{a - \frac{2}{3}} = \dfrac{a^3}{a - \frac{2}{3}}$

(ii) $f'(x) = 3x^2, \ f'(a) = 3a^2$

(iii) $3a^2 = \dfrac{a^3}{a - \frac{2}{3}} \Leftrightarrow 3a^2\left(a - \dfrac{2}{3}\right) = a^3 \Leftrightarrow 3a^3 - 2a^2 = a^3 \Leftrightarrow 2a^3 - 2a^2 = 0$

$\Leftrightarrow 2a^2(a - 1) = 0 \Leftrightarrow a = 1$

(b) Area $= \int_{-2}^{k} (x^3 - 3x + 2)\mathrm{d}x = \left[\dfrac{1}{4}x^4 - \dfrac{3}{2}x^2 + 2x\right]_{-2}^{k}$

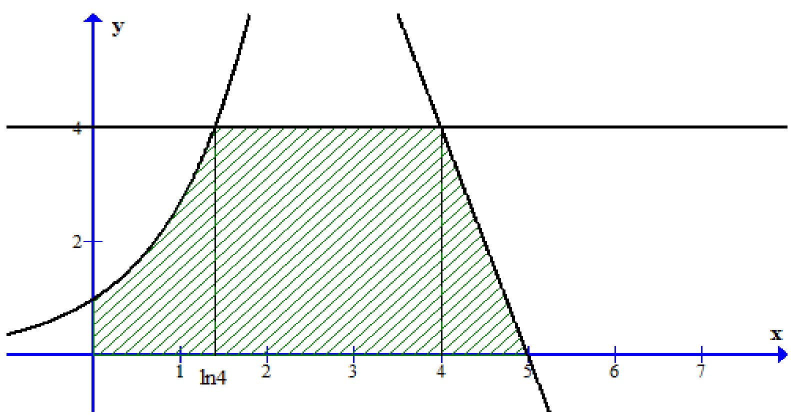
$\left(\dfrac{1}{4}k^4 - \dfrac{3}{2}k^2 + 2k\right) - (4 - 6 - 4) = \dfrac{1}{4}k^4 - \dfrac{3}{2}k^2 + 2k + 6$

$\dfrac{1}{4}k^4 - \dfrac{3}{2}k^2 + 2k + 6 = 2k + 4$

$\dfrac{1}{4}k^4 - \dfrac{3}{2}k^2 + 2 = 0$

$k^4 - 6k^2 + 8 = 0$

51. (a)



y-intercept: (0,1), x-intercept: (5,0), points of intersection: (ln4,4), (4,4)

(b) (Notice that the second part is a rectangle and the third part is a triangle)

$\int_0^{\ln 4} e^x dx + 4(4 - \ln 4) + 2 = 21\text{-}4\ln 4 = \ldots = 21 - 4\ln 4$

(c) (Notice that the second part is a cylinder and the third part is a cone)

$\pi\int_0^{\ln 4} e^{2x} dx + \pi 4^2(4 - \ln 4) + \dfrac{1}{3}\pi 4^2 = \ldots = \left(\dfrac{461}{6} - 16\ln 4\right)\pi$

(d) $\pi\int_0^4 [(5 - y/4)^2 - (\ln y)^2]dy$